KCC-theory and geometry of the Rikitake system

2007 J. Phys. A: Math. Theor. 40 2755

(http://iopscience.iop.org/1751-8121/40/11/011)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 2755–2772 doi:10.1088/1751-8113/40/11/011

KCC-theory and geometry of the Rikitake system

T Yajima and H Nagahama

Department of Geoenvironmental Sciences, Graduate School of Science, Tohoku University
Aoba-ku, Sendai 980-8578, Japan

E-mail: yajima@dges.tohoku.ac.jp

Received 16 September 2006, in final form 28 January 2007
Published 28 February 2007
Online at stacks.iop.org/JPhysA/40/2755

Abstract
The Earth's magnetic field undergoes aperiodical reversals. These can be explained by a simple two-disc dynamo system (Rikitake system). In this paper, the Rikitake system is studied based on a differential geometry (theory of Kosambi–Cartan–Chern). The electrical and mechanical equations of motion are derived from Faraday's law as well as from magnetohydrodynamic equations. From the geometric theory, the solution of the Rikitake system can be regarded as a trajectory on the tangent bundle. Accordingly, there exist five geometrical invariants in the Rikitake system. The third invariant as a torsion tensor can be expressed by mutual-inductances as a result of electrical and mechanical interactions which cause the aperiodic magnetic reversal. This aperiodic behaviour corresponds to a magnetohydrodynamic turbulent motion by a topological invariant such as Chern–Simons number which expresses the interaction between the toroidal and poloidal currents. This Rikitake system is equivalent to other nonlinear dynamical systems. Thus, chaotic behaviours of various nonlinear dynamical systems can be uniformly investigated by the five geometrical invariants and the topological invariant (the Chern–Simons number).

PACS numbers: 02.40.Ky, 05.45.−a, 91.25.Cw, 02.40.−k, 52.30.Cv

1. Introduction

It is known that the geomagnetic field has undergone aperiodic reversals till now. The origin of the magnetic field can be explained by a simple one-disc dynamo system (Faraday disc) [11]. The motion of electric charges in the Faraday disc has already been discussed [10, 29, 30]. However, the aperiodic reversal of the magnetic field cannot occur in the one-disc dynamo system. Therefore, a model for the reversal of the geomagnetic field has been proposed by Rikitake [33]. This model is a simple two-disc dynamo system (Rikitake system). Although the Rikitake system is hard to relate correctly to the real geomagnetic phenomena, the Rikitake system is not an oversimplified geophysical model. For example, the one-disc dynamo system is totally regarded as a single electromechanical system as a whole and does not behave chaotically [1]. On the other hand, the Rikitake system is a combination of two electromechanical systems, because of the addition of another disc system. As a result, the Rikitake system shows chaotic magnetic field reversals.

The Rikitake system has been discussed by many researchers, from various view points. The chaotic magnetic reversals have been discussed by numerical and/or analytical approaches [1, 14, 18, 19, 31]. Mathematically, algebraic geometric methods were applied to find a constant of motion for the Rikitake system [24, 25]. In spite of these, two problems have not been discussed so far.

One is a derivation of the equations of motion. In the previous studies, the equations of motion have been given *a priori* and the focus was on the behaviour of the solution. Therefore, in order to understand the electromechanical structure of the Rikitake system, the equations of motion should be derived by electromagnetic laws.

The other problem is a geometrization of the Rikitake system. In general, the laws of physics should be expressible by geometrical relationships [27]. For example, physical phenomena in Lagrangian mechanics are described by a system of second-order differential equations (Euler–Lagrange equations). Geometrically, the Euler–Lagrange equations are equivalent to geodesic equations (semispray on tangent bundle) [3]. However, the above previous studies have regarded the equations of motion as a system of first-order differential equations and they have not been expressed geometrically yet. Therefore, in order to obtain geometrical expressions, the Rikitake system should be regarded as a system of second-order differential equations.

Geometrically, the second-order differential equations of the Rikitake system can be investigated by the general path-space theory of Kosambi–Cartan–Chern (KCC-theory) in Finsler space (Kosambi [22], Cartan [12], Chern [13]). The KCC-theory is a differential geometric theory of the variational equations for the deviation of whole trajectories to nearby ones. From the KCC-theory, five geometrical invariants are obtained. The second invariant gives the Jacobi stability. The third invariant expresses a torsion tensor. The KCC-theory has been applied to the field of electrical engineering. For example, in the theory of electrical machinery, the KCC-theory has been applied to the unified electromechanical system [23]. The variational equations (hunting equations) have been derived in order to investigate the behaviour of trajectories which are perturbed by the operation of the machine. The stability of an airplane in flight has also been discussed [21] by the KCC-theory and Schouten's film space [35]. Similarly, the Rikitake system can be regarded as an electric machinery and so the KCC-theory can be applied to such a system. The geometrical objects then express the chaotic behaviour of the Rikitake system.

This chaotic behaviour of the Rikitake system is related to the motion of a magnetohydrodynamic fluid. In dynamo theory, the initial magnetic field is dragged along by the fluid motion and then sheared by the differential rotation. This winding motion known as ω-effect creates a new magnetic field. This magnetic field then is twisted by a helical turbulent motion due to the α-effect [32]. The turbulent α-effect is related to a topological invariant called the magnetic helicity which is a measure of the twistedness of the magnetic field [8]. In the Rikitake system, the α-effect is expressed by the angular velocity of the rotating disc [28]. Therefore, the relation between chaotic behaviour and magnetohydrodynamic motion can be investigated by the topological invariant.

In this paper, the chaotic behaviour of the nonlinear dynamical system is expressed by the geometrical and topological invariants. This paper consists of five sections. In section 2, the KCC-theory is reviewed briefly. In section 3, the equations of motion for the Rikitake system are derived based on Faraday's law. In section 4, the KCC-theory is applied to the Rikitake system. Geometrical invariants of the Rikitake system are obtained. In section 5, the relationship between the behaviour of the magnetic fields reversal and the geometrical objects is discussed. Besides, the chaotic behaviour of the Rikitake system is also compared with magnetohydrodynamic motion. Finally, it is pointed out that other nonlinear dynamical systems can also be analysed by the geometrical and topological invariants.

2. KCC-theory and Jacobi equation

In this section, the geometrical background of the system of second-order differential equations is introduced. Throughout this paper, Einstein's summation convention is used. Moreover, Latin indices $i, j, k, \ldots$ run from 1 to n.

2.1. Semispray and a constant of motion

Let $\mathscr{M}$ be a real smooth n-dimensional manifold and $T\mathscr{M}$ be its tangent bundle. Let $(x^i) = (x^1, \ldots, x^n)$,

$$(y^i) = \left(\frac{\mathrm{d}x^i}{\mathrm{d}t}\right) = \left(\frac{\mathrm{d}x^1}{\mathrm{d}t}, \frac{\mathrm{d}x^2}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}x^n}{\mathrm{d}t}\right) \tag{1}$$

and time t be $2n+1$ local coordinates (t, x^i, y^i) on an open connected subset U of the Euclidean $(2n+1)$-dimensional space $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n$. The time t is regarded as an absolute invariant. Therefore, the change of coordinates will be

$$\tilde{t} = t, \qquad \tilde{x}^i = \tilde{x}^i(x^1, x^2, \ldots, x^n). \tag{2}$$

Generally, the equations of motion in Finsler space are given by the Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = \mathscr{F}_i, \tag{3}$$

where the scalar function L is the Lagrangian and $\mathscr{F}_i$ is an external force. The triple $(\mathscr{M}, L, \mathscr{F}_i)$ is called the Finslerian mechanical system [26]. For a regular Lagrangian L, the Euler–Lagrange equations (3) are equivalent to a system of second-order differential equations:

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + 2G^i(x^j, y^j, t) = 0, \tag{4}$$

where the function $G^i(x^j, y^j, t)$ is smooth in a neighbourhood of some initial conditions $\left(x_0^i, y_0^i, t_0\right) \in U$. Moreover, the system of second-order differential equation is equivalent to a vector field (semispray) S which determines a nonlinear connection N_j^i [3]:

$$S = y^i\frac{\partial}{\partial x^i} - 2G^i(x^j, y^j, t)\frac{\partial}{\partial y^i}, \tag{5}$$

$$N_j^i = \frac{\partial G^i}{\partial y^j}. \tag{6}$$

Such a dynamical system as the semispray does not behave chaotically when it has a constant of motion. The following theorem is known [36, 37]:

Theorem 1. *For a vector field X, assume that* $[X, S] = \mathcal{F}S$, *where* $\mathcal{F}$ *is a real valued function and* $[,]$ *is a Lie bracket. Then, the relation holds*

$$L_S(\mathcal{F} + \mathrm{div}_\Omega X) = -\mathcal{F}\,\mathrm{div}_\Omega S + L_X\,\mathrm{div}_\Omega S, \tag{7}$$

where $\mathrm{div}_\Omega S$ *denotes the divergence of S with respect to the volume form* Ω. $L_S\Omega = (\mathrm{div}_\Omega S)\Omega$ *is a Lie derivative of* Ω *along the vector field S.*

When $\mathcal{F} = 0$ and $\mathrm{div}_\Omega S$ = constant, the right-hand side of (7) is equal to zero. Therefore, $L_S(\mathcal{F} + \mathrm{div}_\Omega X) = 0$ and the divergence of X does not change. The function $\mathrm{div}_\Omega X$ is a constant of motion and X is called the symmetry generator [36, 37].

2.2. Geometric theory of a system of second-order differential equations

In the following, the geometric theory (KCC-theory) is briefly reviewed based on the notations [3–5].

Let us consider a system of second-order differential equations (4). Under the non-singular coordinates transformation (2), the KCC-covariant differential of a vector field $\xi^i(t)$ on the open subset $U \subseteq \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n$ is defined as follows:

$$\frac{\mathrm{D}\xi^i}{\mathrm{d}t} = \frac{\mathrm{d}\xi^i}{\mathrm{d}t} + N^i_j\xi^j. \tag{8}$$

When we put $\xi^i = y^i$, the covariant differential becomes

$$\frac{\mathrm{D}y^i}{\mathrm{d}t} = -\epsilon^i \equiv N^i_j y^j - 2G^i, \tag{9}$$

where ϵ^i is a contravariant vector field on U and is called the first KCC-invariant.

Then, consider that the trajectory $x^i(t)$ of the system (4) is varied into nearby ones according to

$$\bar{x}^i(t) = x^i(t) + \xi^i(t)\eta, \tag{10}$$

where η denotes a parameter with $|\eta|$ small and the components of contravariant vector $\xi^i(t)$ are defined along a curve $x^i = x^i(t)$. Substituting (10) into (4) and taking the limit $\eta \to 0$, one gets the variational equations

$$\frac{\mathrm{d}^2\xi^i}{\mathrm{d}t^2} + 2N^i_l\frac{\mathrm{d}\xi^l}{\mathrm{d}t} + 2\frac{\partial G^i}{\partial x^l}\xi^l = 0. \tag{11}$$

Using the KCC-covariant differential (8), one rewrites (11) in the covariant form

$$\frac{\mathrm{D}^2\xi^i}{\mathrm{d}t^2} + P^i_l\xi^l = 0, \tag{12}$$

where

$$P^i_j = 2\frac{\partial G^i}{\partial x^j} + 2G^l G^i_{jl} - y^l\frac{\partial N^i_j}{\partial x^l} - N^i_l N^l_j - \frac{\partial N^i_j}{\partial t}. \tag{13}$$

Here, the $G^i_{jk} \equiv \partial N^i_j/\partial y^k$ is a kind of Finsler connection (Berwald connection) [7]. This variational equation (12) is called the Jacobi equation or 'hunting equation' in the field of engineering [23]. The P^i_j is called the second KCC-invariant or deviation curvature tensor and gives the stability of whole trajectories from the following theorem [6, 34]:

Theorem 2. *The trajectories of system (4) are Jacobi stable if and only if the real parts of the eigenvalues of the deviation curvature tensor* P^i_j *are strictly negative everywhere, and Jacobi unstable otherwise.*

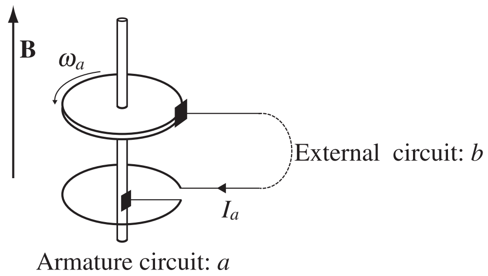

Figure 1. One-disc dynamo system (modified from figure in [11]).

The third, fourth and fifth invariants of the system (4) are given by

$$P^i_{jk} \equiv \frac{1}{3}\left(\frac{\partial P^i_j}{\partial y^k} - \frac{\partial P^i_k}{\partial y^j}\right), \qquad P^i_{jkl} \equiv \frac{\partial P^i_{jk}}{\partial y^l}, \qquad D^i_{jkl} \equiv \frac{\partial G^i_{jk}}{\partial y^l}. \tag{14}$$

Because of the skew symmetry of lower indices j and k, the third invariant is regarded as torsion tensor. The fourth invariant is the Riemann–Christoffel curvature tensor and the fifth invariant D^i_{jkl} is a kind of curvature tensor (Douglas tensor) [15].

Generally, in the Berwald space, there exist the two curvature tensors P^i_{jkl}, D^i_{jkl} and the one torsion tensor P^i_{jk} [5]. Therefore, these geometrical objects express the geometrical properties of the system of second-order differential equations.

3. Fundamental equations for a dynamo system

In this section, in the case of a one-disc dynamo system, the equations of motion are derived first from Faraday's law. Consequently, the equations of motion for the Rikitake system are obtained based on the above derivation.

3.1. One-disc dynamo system

In a one-disc dynamo system, the rotating disc has two types of coordinate frames. One is a laboratory frame or fixed frame in which the circuit is at rest. The other is a rotating observer frame. Let us consider one-disc dynamo system from the laboratory frame. In this case, the total circuit C is fixed to the disc and consists of the armature circuit a and the external circuit b (figure 1). We denote the variables in the one-disc dynamo system with indices a and b that correspond to armature circuit a and external circuit b, respectively.

A cylindrical coordinate system (r, θ, z) and its basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ are used in order to derive the equations of motion. Here, direction of the radius of disc is r, direction of the rotating angle of disc is θ and direction of the normal to the disc is z. Radius of the disc is denoted by h. In a one-disc dynamo system, the disc rotates about its axis with angular velocity $\omega = \omega_a \mathbf{e}_z$ which is the differentiation of the angle with respect to time; $\omega_a = \mathrm{d}\theta/\mathrm{d}t$. Then, the velocity of the conductor is given by $\mathbf{v} = r\omega_a \mathbf{e}_\theta$. A path of current I_a between its rim and axle is provided by the wire twisted as shown in a loop around the axle. The electric current is given by the time differential of the charge; $I_a = \mathrm{d}q/\mathrm{d}t$. The current I_a generates a magnetic flux density $\mathbf{B} = B_z \mathbf{e}_z$ across the disc. The electric field is radially directed, i.e. $\mathbf{E} = E_r \mathbf{e}_r$.

The electrical equation can be derived from Faraday's law. In the laboratory frame, Faraday's law for the total circuit C can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} = -\oint_C \mathbf{E}\cdot \mathrm{d}\mathbf{l}, \tag{15}$$

where the $\mathrm{d}\mathbf{l}$ and $\mathrm{d}\mathbf{S}$ express an infinitesimal length of the circuit and an element of the disc, respectively. The contour integral is split into two parts, i.e. one is the armature circuit a and the other is the external circuit b [39]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} = -\int_a \mathbf{E}\cdot \mathrm{d}\mathbf{l} - \int_b \mathbf{E}\cdot \mathrm{d}\mathbf{l}. \tag{16}$$

The second term on the right-hand side of equation (16) is the external voltage V_b. In the disc dynamo model, there is no external voltage, and so $V_b = 0$.

On the other hand, the first term on the right-hand side of equation (16) is the armature voltage. This term can be rewritten by using Ohm's law in the laboratory frame: $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B})$, where σ is a conductivity of the system and $\mathbf{J}$ is the current density. By Ohm's law, the integral of armature circuit is obtained [39]:

$$-\int_a \mathbf{E}\cdot \mathrm{d}\mathbf{l} = -\int_a \frac{\mathbf{J}}{\sigma}\cdot \mathrm{d}\mathbf{l} + \int_a (\mathbf{v}\times\mathbf{B})\cdot \mathrm{d}\mathbf{l}. \tag{17}$$

Thus, the Faraday's law is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} = -\int_a \frac{\mathbf{J}}{\sigma}\cdot \mathrm{d}\mathbf{l} + \int_a (\mathbf{v}\times\mathbf{B})\cdot \mathrm{d}\mathbf{l}. \tag{18}$$

The first term on the right-hand side of equation (18) is the voltage V_a across the armature resistance R_1^1: $V_a = -R_1^1 I_a$, where the upper index 1 of R_1^1 expresses an electrical system and the lower index 1 expresses the electric current I_a.

On the other hand, the second term on the right-hand side of equation (18) represents the electromotive force arising from the rotation ω_a (speed voltage [39]). This second term can be integrated from the centre to the periphery of disc

$$\begin{aligned}\int_0^h (\mathbf{v}\times\mathbf{B})\cdot \mathrm{d}\mathbf{l} &= \frac{1}{2}h^2 B_z \omega_a \\ &= M_{12}^1 I_a \omega_a,\end{aligned} \tag{19}$$

where $M_{12}^1 I_a \equiv h^2 B_z/2$ because the magnetic flux is proportional to current. The coefficient M_{12}^1 is the mutual-inductance between the variables with lower indices 1 and 2, i.e. the current I_a and the angular velocity ω_a with respect to the upper index 1 of the electrical system.

Finally, since the magnetic flux through the coil is only due to the current I_a, the left-hand side of (18) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} = \frac{\mathrm{d}\left(L_1^1 I_a\right)}{\mathrm{d}t} = L_1^1 \frac{\mathrm{d}I_a}{\mathrm{d}t}, \tag{20}$$

where L_1^1 is the self-inductance between the electrical system expressed by the upper index 1 and the current I_a expressed by the lower index 1. Thus from (19) and (20), the following proposition is obtained:

Proposition 1. *The electrical equation of motion for the one-disc dynamo system can be expressed by*

$$L_1^1 \frac{\mathrm{d}I_a}{\mathrm{d}t} = -R_1^1 I_a + M_{12}^1 I_a \omega_a, \tag{21}$$

where I_a is the current flowing in the circuit, ω_a is the angular velocity of disc, R_1^1 is the armature resistance, M_{12}^1 is the mutual-inductance between coil and disc, and L_1^1 is the self-inductance.

Next, a mechanical equation is derived from the Navier–Stokes equation

$$\rho\frac{\partial\tilde{\mathbf{v}}}{\partial t}+\rho(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}=-\nabla p-\nabla U+\mathbf{F}^e, \tag{22}$$

where $\tilde{\mathbf{v}}$ is the velocity of fluid motion, ρ is density, p is pressure, U is potential and $\mathbf{F}^e$ is an external force including the electrical force. Then, suppose that the mechanical motion is induced by this external force alone:

$$p=0,\qquad U=0,\qquad \mathbf{F}^e=\mathbf{J}\times\mathbf{B}+\mathbf{F}'. \tag{23}$$

Here, $\mathbf{J}\times\mathbf{B}=-I_aB_z\mathbf{e}_\theta/2\pi r$ is the Lorentz force. $\mathbf{F}'=F'_\theta\mathbf{e}_\theta$ is the mechanical force acting in the tangential direction of the disc. The velocity of the fluid motion $\tilde{\mathbf{v}}$ is regarded as the angular velocity of the disc $\mathbf{v}$. In this case, $\rho(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}$ vanishes because the velocity $\tilde{\mathbf{v}}$ is given by $\tilde{\mathbf{v}}=\mathbf{v}=r\omega_a\mathbf{e}_\theta$. Thus, equation (22) is reduced to

$$\rho r\frac{\mathrm{d}\omega_a}{\mathrm{d}t}=-\frac{B_z}{2\pi r}I_a+F'_\theta. \tag{24}$$

In order to find the total torque, both sides of equation (24) are multiplied by the radius of disc $\mathbf{r}=r\mathbf{e}_r$ and integrated throughout the volume of the disc:

$$\int_S\rho r^2\,\mathrm{d}S\frac{\mathrm{d}\omega_a}{\mathrm{d}t}=-\int_S r\frac{B_z}{2\pi r}I_a\,\mathrm{d}S+\int_S rF'_\theta\,\mathrm{d}S. \tag{25}$$

The integrand on the left-hand side of equation (25) is the inertial moment of disc. The first and second terms on the right-hand side of equation (25) express the mutual-inductance and driving couple, respectively. Thus, the mechanical equation of motion is given by

$$J_2^2\frac{\mathrm{d}\omega_a}{\mathrm{d}t}=-M_{11}^2(I_a)^2+F^2, \tag{26}$$

where the relation $M_{11}^2I_a\equiv h^2B_z/2$ is used. The constants J_2^2 and F^2 are regarded as the inertial moment of the disc and the driving couple, respectively. The upper index 2 of the coefficients denotes the mechanical part of the one-disc dynamo system. Thus, the following proposition can be obtained:

Proposition 2. *The mechanical equation of motion for the one-disc dynamo system can be expressed by*

$$J_2^2\frac{\mathrm{d}\omega_a}{\mathrm{d}t}=F^2-M_{11}^2(I_a)^2, \tag{27}$$

where J_2^2 and F^2 are the inertial moment of the disc and the driving couple, respectively.

Equations (21) and (27) are the equations of motion for the one-disc dynamo system.

3.2. The equations of motion for the Rikitake system

Rikitake [33] has considered a two-disc dynamo system as system I and system II (figure 2). The rotation of disc I (ω^1) in initial magnetic field induces a current of system I (I^1) which produces a magnetic field $\mathbf{B}_2$ through disc II. The interaction between the magnetic field $\mathbf{B}_2$ and rotation of disc II (ω^2) induces a current of system II (I^2) which produces a magnetic field $\mathbf{B}_1$ through disc I. The interaction between $\mathbf{B}_1$ and rotation of disc I regenerates the current I^1 which reinforces the initial magnetic field. This feedback system maintains the

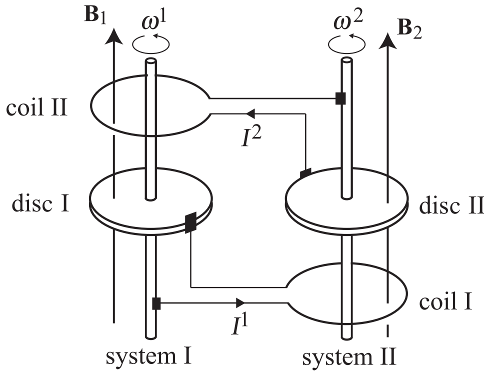

Figure 2. The Rikitake system (modified from a figure in [33]).

magnetic field of the Rikitake system. The dynamo process is given by the coupled one-disc dynamo systems. Hence, the equations of motion for the Rikitake system are obtained from propositions 1 and 2 [33]:

Theorem 3. *In the Rikitake system, the equations of motion are given by*

$$\begin{aligned} L_1^1 \frac{\mathrm{d}I^1}{\mathrm{d}t} + R_1^1 I^1 &= M_{23}^1 I^2 \omega^1, & L_2^2 \frac{\mathrm{d}I^2}{\mathrm{d}t} + R_2^2 I^2 &= M_{14}^2 I^1 \omega^2, \\ J_3^3 \frac{\mathrm{d}\omega^1}{\mathrm{d}t} &= F^3 - M_{12}^3 I^1 I^2, & J_4^4 \frac{\mathrm{d}\omega^2}{\mathrm{d}t} &= F^4 - M_{12}^4 I^1 I^2, \end{aligned} \tag{28}$$

where the variables, I^i and ω^i, represent the current and angular velocity with the subscripts corresponding to the system number I or II, respectively. The coefficients are all positive constants. The $\left(L_1^1, L_2^2\right)$, $\left(R_1^1, R_2^2\right)$, $\left(J_3^3, J_4^4\right)$ and (F^3, F^4) are the self-inductances, resistances, moments of inertia and couples, respectively. The indices of the coefficients 1 and 2 express the electrical part of system I and II, respectively. The indices of the coefficients 3 and 4 express the mechanical part of system I and II, respectively. M_{jk}^i is the mutual-inductance of system I and system II. For example, M_{23}^1 is the interaction between variables with the lower indices, the current I^2 and the angular velocity ω^1, with respect to variable with upper index of the current I^1.

With the aid of symmetry,

$$L_1^1 = L_2^2, \qquad M_{23}^1 = M_{14}^2 = M_{12}^3 = M_{12}^4, \qquad J_3^3 = J_4^4, \qquad R_1^1 = R_2^2, \qquad F^3 = F^4, \tag{29}$$

the equations of motion (28) can be written in dimensionless form (not tensor form) as

$$\frac{\mathrm{d}I^1}{\mathrm{d}t} = -\mu I^1 + I^2\omega^1, \qquad \frac{\mathrm{d}I^2}{\mathrm{d}t} = -\mu I^2 + I^1\omega^2, \qquad \frac{\mathrm{d}\omega^1}{\mathrm{d}t} = 1 - I^1 I^2, \tag{30}$$

where $\omega^2 = \omega^1 - \nu$ and $\nu = \mu\{(k)^2 - (k)^{-2}\}$. The μ and k are constants.

These equations for the Rikitake system are equivalent to equations of motion in the case of theory of magnetohydrodynamic dynamo. In the same way of the disc dynamo system, Faraday's and Ohm's laws give the induction equations in the case of magnetohydrodynamic dynamo theory [28]:

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{\sigma} \nabla \times (\nabla \times \mathbf{B}) + \nabla \times (\tilde{\mathbf{v}} \times \mathbf{B}). \tag{31}$$

The resistance term $R^i_j I^j$ in (28) corresponds to the diffusion term in the first one on the right-hand side of equation (31). The electromotive force $M^i_{jk} I^j \omega^k$ generated by the rotating disc in (28) is equivalent to the electromotive force due to the interaction between the fluid motion and magnetic flux in the second term on the right-hand side of equation (31). Therefore, the electrical equations of motion correspond to the induction equations.

Moreover, the variables and terms in the equations of motion (28) express characteristics of the magnetohydrodynamic dynamo action [28]. The current I^1 can be regarded as the total toroidal current. The angular velocity ω^2 represents the mean differential rotation in the core. This differential rotation is generated by the driving force F^4. Therefore, the term $M^2_{14} I^1 \omega^2$ represents the production of toroidal magnetic field due to the ω-effect. On the other hand, the current I^2 can be regarded as the total poloidal current. The angular velocity ω^1 represents a measure of the intensity of the α-effect which is generated by the driving force F^3. Therefore, the term $M^1_{23} I^2 \omega^1$ represents the production of the poloidal magnetic field due to the α-effect. These ω- and α-effects reinforce the original magnetic field. Hence, these feedback processes of the Rikitake system correspond to the $\alpha\omega$-dynamo system.

4. Geometrical description of the Rikitake system

In this section, the equations of motion (28) are regarded as a semispray on the tangent bundle. Then, from the KCC-theory, geometrical invariants of the Rikitake system are obtained. In the following, the Latin indices $i, j, k, \ldots$ run from 1 to 4.

4.1. The semispray of the Rikitake system

Let $(x^i) = (q^1, q^2, \theta^1, \theta^2)$ be the natural coordinates. The coordinates x^1 and x^2 are interpreted as the electric charges q^1 and q^2 in the system I and II, respectively. On the other hand, the x^3 and x^4 are interpreted as the angles of the rotating discs θ^1 and θ^2 in the system I and II, respectively. Let (x^i, y^i) denote natural coordinates in a local chart of the tangent bundle, where $y^i = (I^1, I^2, \omega^1, \omega^2)$.

In order to obtain a geometrical description of the Rikitake system, the equations of motion (28) need to be brought into the form of a semispray (5). In the case of $i = 1$, the equation of motion for the electrical part of system I,

$$I^1 = \frac{\mathrm{d}q^1}{\mathrm{d}t}, \qquad L^1_1 \frac{\mathrm{d}I^1}{\mathrm{d}t} + R^1_1 I^1 = M^1_{23} I^2 \omega^1 \tag{32}$$

can be regarded as

$$y^1 = \frac{\mathrm{d}x^1}{\mathrm{d}t}, \qquad \frac{\mathrm{d}y^1}{\mathrm{d}t} + 2G^1(t, x^j, y^j) = 0. \tag{33}$$

By comparing both equations (32) and (33), the coefficient G^1 is

$$G^1(t, x^j, y^j) = -\frac{M^1_{23}}{2L^1_1} y^2 y^3 + \frac{R^1_1}{2L^1_1} y^1. \tag{34}$$

From the definition $N^i_j = \partial G^i / \partial y^j$, the nonlinear connection N^1_j is

$$N^1_j = -\frac{M^1_{23}}{2L^1_1} \left(\delta^2_j y^3 + \delta^3_j y^2 \right) + \frac{R^1_1}{2L^1_1} \delta^1_j. \tag{35}$$

Moreover, from N^1_j, the Berwald connection G^1_{jk} is

$$G^1_{jk} = -\frac{M^1_{23}}{2L^1_1} \left(\delta^2_k \delta^3_j + \delta^3_k \delta^2_j \right). \tag{36}$$

Therefore, the components of the nonlinear connection N^1_j and the Berwald connection G^1_{jk} can be expressed as

$$N^1_1 = \frac{R^1_1}{2L^1_1}, \qquad N^1_2 = -\frac{M^1_{23}}{2L^1_1} y^3, \qquad N^1_3 = -\frac{M^1_{23}}{2L^1_1} y^2, \qquad G^1_{23} = G^1_{32} = -\frac{M^1_{23}}{2L^1_1}.$$

Thus, the equations of motion (32) or (33) can be rewritten as follows:

$$y^1 = \frac{\mathrm{d}x^1}{\mathrm{d}t}, \qquad \frac{\mathrm{d}y^1}{\mathrm{d}t} + 2G^1_{23} y^2 y^3 + 2N^1_1 y^1 = 0. \tag{37}$$

Similarly, for $i = 2, 3, 4$, the equations of motion can be rewritten as follows:

$$y^2 = \frac{\mathrm{d}x^2}{\mathrm{d}t}, \qquad \frac{\mathrm{d}y^2}{\mathrm{d}t} + 2G^2_{14} y^1 y^4 + 2N^2_2 y^2 = 0, \tag{38}$$

$$y^3 = \frac{\mathrm{d}x^3}{\mathrm{d}t}, \qquad \frac{\mathrm{d}y^3}{\mathrm{d}t} + 2G^3_{12} y^1 y^2 = f^3, \tag{39}$$

$$y^4 = \frac{\mathrm{d}x^4}{\mathrm{d}t}, \qquad \frac{\mathrm{d}y^4}{\mathrm{d}t} + 2G^4_{12} y^1 y^2 = f^4, \tag{40}$$

where f^3 and f^4 are defined as

$$f^3 \equiv \frac{F^3}{J^3_3} \qquad \text{and} \qquad f^4 \equiv \frac{F^4}{J^4_4}. \tag{41}$$

As a result, the equations of motion (28) and the semispray with the nonlinear connection are

$$y^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}, \qquad \frac{\mathrm{d}y^i}{\mathrm{d}t} = -G^i_{jk} y^j y^k + \gamma^i_j y^j + f^i, \tag{42}$$

$$S = y^i \frac{\partial}{\partial x^i} - \left(G^i_{jk} y^j y^k - \gamma^i_j y^j - f^i\right) \frac{\partial}{\partial y^i}, \tag{43}$$

$$N^i_j = G^i_{jk} y^k - \frac{1}{2}\gamma^i_j. \tag{44}$$

Here, the coefficients G^i_{jk}, γ^i_j and f^i are

$$\begin{cases} G^1_{23} = G^1_{32} = -\dfrac{M^1_{23}}{2L^1_1}, G^2_{14} = G^2_{41} = -\dfrac{M^2_{14}}{2L^2_2}, \\ G^3_{12} = G^3_{21} = \dfrac{M^3_{12}}{2J^3_3}, G^4_{12} = G^4_{21} = \dfrac{M^4_{12}}{2J^4_4}, \end{cases} \tag{45}$$

$$\gamma^1_1 = -2N^1_1 = -\frac{R^1_1}{L^1_1}, \qquad \gamma^2_2 = -2N^2_2 = -\frac{R^2_2}{L^2_2}, \qquad \gamma^3_3 = \gamma^4_4 = 0, \tag{46}$$

$$f^1 = f^2 = 0, \qquad f^3 = \frac{F^3}{J^3_3}, \qquad f^4 = \frac{F^4}{J^4_4}. \tag{47}$$

The tensor expression (45) tells us that the constant Berwald connection G^i_{jk} plays an important role in the interaction between the electrical and mechanical systems. For example, G^1_{23} denotes the interaction between the poloidal current I^2 and the angular velocity ω^1 which is the intensity of α-effect. The nonlinear connection is given by a base connection in Finsler space; $\delta y^i/\mathrm{d}t = \mathrm{d}y^i/\mathrm{d}t + N^i_j y^j$. Therefore, the nonlinear connection expresses the interaction between the (y^i)-field and (y^j)-field.

The Rikitake system does not behave chaotically when there is a constant of motion for the semispray (43). The non-chaotic behaviour is given by the condition that the discs of the Rikitake system rotate with the similar angular velocity, that is $\omega^1 = \omega^2$. In addition, the coefficients of the Rikitake system should satisfy the symmetry condition (29). Under the conditions of non-chaotic behaviour, there exists a vector field called the symmetric generator [37]

$$X = \{(y^1)^2 - (y^2)^2\} \exp(2LRt) \frac{\partial}{\partial t}, \tag{48}$$

where $L \equiv L^1_1 = L^2_2$ and $R \equiv R^1_1 = R^2_2$. The symmetric generator satisfies the Lie bracket relation $[X, S] = \mathcal{F}X = 0$ given in theorem 1, i.e. the real valued function $\mathcal{F}$ is equal to zero. On the other hand, the divergence of semispray (43) is $\mathrm{div}_\Omega S = -2LR =$ constant. Therefore, the Lie derivative of the semispray vanishes, i.e. $L_X \,\mathrm{div}_\Omega S = 0$. Consequently, from equation (7), the Lie derivative for the constant of motion $Y \equiv \mathrm{div}_\Omega X$ vanishes [37], i.e. $L_S Y = 0$. The existence of the constant of motion Y means that trajectories of the Rikitake system are constrained on a certain plane in the phase space. In other words, the Rikitake system with the constant of motion becomes a holonomic system and hence does not behave chaotically.

4.2. Application of the KCC-theory to the Rikitake system in film space

The chaotic behaviour of the Rikitake system can be geometrically expressed by the KCC-theory. The behaviour of the Rikitake system can be described by the world line in (4 + 1)-dimensional space-time, whose four-dimensional projection is observed as the usual trajectory. For Greek indices $\alpha, \beta, \gamma = 0, 1, 2, 3, 4$, $x^0 = t \in \mathbf{R}$ and $\mathrm{d}x^0/\mathrm{d}t = 1$, the connection coefficients are

$$\begin{aligned} &\Gamma^\alpha_{\beta\gamma} = \Gamma^i_{jk} + \Gamma^i_{j0} + \Gamma^i_{0j} + \Gamma^i_{00} + \Gamma^0_{jk}, \qquad && \Gamma^i_{jk} = G^i_{jk}, \\ &\Gamma^i_{j0} = -\tfrac{1}{2}\gamma^i_j = \Gamma^i_{0j}, \qquad \Gamma^i_{00} = -f^i, && \Gamma^0_{\beta\gamma} = 0. \end{aligned} \tag{49}$$

Then, the system (42) is

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = y^\alpha, \qquad \frac{\mathrm{d}y^\alpha}{\mathrm{d}t} = -\Gamma^\alpha_{\beta\gamma} y^\beta y^\gamma. \tag{50}$$

Moreover, equation (50) can be rewritten in the Pfaffian form as

$$\mathrm{d}x^\alpha - y^\alpha\,\mathrm{d}t = 0, \qquad \mathrm{d}y^\alpha + \Gamma^\alpha_\gamma y^\gamma = 0, \tag{51}$$

where Γ^α_γ is a $(4 + 1) \times (4 + 1)$ matrix valued connection 1-form: $\Gamma = \left(\Gamma^\alpha_\gamma\right) = \left(\Gamma^\alpha_{\beta\gamma}\,\mathrm{d}x^\beta\right)$. The components of the connection 1-form Γ can be expressed by

$$\left(\Gamma^\alpha_\gamma\right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \Gamma^1_0 & 0 & \Gamma^1_2 & \Gamma^1_3 & 0 \\ \Gamma^2_0 & \Gamma^2_1 & 0 & 0 & \Gamma^2_4 \\ \Gamma^3_0 & \Gamma^3_1 & \Gamma^3_2 & 0 & 0 \\ \Gamma^4_0 & \Gamma^4_1 & \Gamma^4_2 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \Gamma^1_{10}\,\mathrm{d}x^1 & 0 & \Gamma^1_{32}\,\mathrm{d}x^3 & \Gamma^1_{23}\,\mathrm{d}x^2 & 0 \\ \Gamma^2_{20}\,\mathrm{d}x^2 & \Gamma^2_{41}\,\mathrm{d}x^4 & 0 & 0 & \Gamma^2_{14}\,\mathrm{d}x^1 \\ \Gamma^3_{00}\,\mathrm{d}x^0 & \Gamma^3_{21}\,\mathrm{d}x^2 & \Gamma^3_{12}\,\mathrm{d}x^1 & 0 & 0 \\ \Gamma^4_{00}\,\mathrm{d}x^0 & \Gamma^4_{21}\,\mathrm{d}x^2 & \Gamma^4_{12}\,\mathrm{d}x^1 & 0 & 0 \end{pmatrix}. \tag{52}$$

The formalization of this extended space called the film space [21, 35] is adopted in view of the effect of geometrization. From these film space notations, the KCC-theory can be applied to the Rikitake system. The geometrical objects are then obtained as follows.

4.3. First invariant

The first invariant of the Rikitake system is given by (9):

$$\epsilon^1 = \frac{R^1_1}{2L^1_1} I^1, \qquad \epsilon^2 = \frac{R^2_2}{2L^2_2} I^2, \qquad \epsilon^3 = -\frac{F^3}{J^3_3}, \qquad \epsilon^4 = -\frac{F^4}{J^4_4}. \tag{53}$$

Thus, the first invariant represents the voltages $R^1_1 I^1$, $R^2_2 I^2$ and the driving couples F^3, F^4, i.e. the first invariant of the Rikitake system is the external force.

4.4. Second invariant and the variational equation of the Rikitake system

Next, let us investigate the Jacobi stability of the Rikitake system. When the trajectories of the charge or angle x^i deviate from the normal paths, i.e. $\bar{x}^i(t) = x^i(t) + \xi^i(t)\eta$, the variational equation (12) gives the behaviour of the deviated trajectories [4]:

$$\frac{D^2\xi^i}{\mathrm{d}t^2} + P^i_{\alpha\beta l} y^\alpha y^\beta \xi^l = 0, \tag{54}$$

where P^i_{jkl}, P^i_{0kl} and P^i_{00l} are

$$P^i_{jkl} = 2\left(\frac{\partial\Gamma^i_{j[k}}{\partial x^{l]}} + \Gamma^m_{j[k}\Gamma^i_{l]m} + N^r_{[k} D^i_{l]rj}\right), \qquad \frac{\partial\Gamma^i_{j[k}}{\partial x^{l]}} \equiv \frac{1}{2}\left(\frac{\partial\Gamma^i_{jk}}{\partial x^l} - \frac{\partial\Gamma^i_{jl}}{\partial x^k}\right), \tag{55}$$

$$P^i_{0kl} = \frac{1}{2}\left(\gamma^i_{l|k} - \gamma^i_{k|l}\right), \qquad \gamma^i_{l|k} \equiv \frac{\partial\gamma^i_l}{\partial x^k} - N^h_k \frac{\partial\gamma^i_l}{\partial y^h} + \gamma^m_l \Gamma^i_{mk} - \gamma^i_m \Gamma^m_{lk}, \tag{56}$$

$$P^i_{00l} = \frac{1}{2}\frac{\partial\gamma^i_l}{\partial t} - \Gamma^i_{ml} f^m - \frac{1}{4}\gamma^i_m \gamma^m_l. \tag{57}$$

In the field of engineering, this variational equation is called the hunting equation introduced by Kron in order to study the stability of electrical machine systems [23].

In case of the Rikitake system, the coefficients (55), (56) and (57) can be calculated as follows. The Douglas tensor D^i_{jkl} as the fifth invariant vanishes because the equations of motion (28) are quadratic differential equations whose coefficients are all constants. This shows that the fourth invariant P^i_{jkl} can be reduced to the usual Riemannian curvature tensor:

$$P^i_{jkl} = 2\Gamma^m_{j[k}\Gamma^i_{l]m}. \tag{58}$$

Moreover, the dissipation terms of (42) can be shown to be P^i_{0kl} and P^i_{00l} as

$$P^i_{0kl} = \gamma^m_{[l}\Gamma^i_{k]m}, \qquad P^i_{00l} = -\Gamma^i_{lh} f^h - \tfrac{1}{4}\gamma^i_m \gamma^m_l. \tag{59}$$

Thus, we can determine the instability of the deviated path from the variational equations:

$$\frac{D^2\xi^r}{dt^2}+\left\{2\Gamma^m_{j[k}\Gamma^r_{l]m}y^jy^k+\left(-\gamma^m_j\Gamma^r_{ml}+\frac{1}{2}\gamma^m_l\Gamma^r_{mj}+\frac{1}{2}\gamma^r_m\Gamma^m_{lj}\right)y^j-\Gamma^r_{lh}f^h-\frac{1}{4}\gamma^r_m\gamma^m_l\right\}\xi^l=0, \tag{60}$$

$$\frac{D^2\xi^s}{dt^2}+\left\{2\Gamma^m_{j[k}\Gamma^s_{l]m}y^jy^k+\left(-\gamma^m_j\Gamma^s_{ml}+\frac{1}{2}\gamma^m_l\Gamma^s_{mj}+\frac{1}{2}\gamma^s_m\Gamma^m_{lj}\right)y^j\right\}\xi^l=0, \tag{61}$$

where the subscripts $i = 1, 2, 3, 4$ are divided into electrical deviations $r = 1, 2$ and mechanical deviations $s = 3, 4$. The Jacobi field equation can also be obtained from the Lie derivative of the connection in film space along the deviation vector ξ^i [40]. The connection can be expressed by the mutual-inductances as the interaction between electrical and mechanical systems. Therefore, the Jacobi equation gives the change in interaction along the deviated direction ξ^i.

4.5. Third invariant

The third invariant is obtained by differentiating P^i_j and P^i_k with respect to y^k and y^j, respectively:

$$P^i_{jk}=2\Gamma^i_{m[j}N^m_{k]}. \tag{62}$$

Because of the existence of the torsion tensor, the trajectories of the Rikitake system are not in a closed loop which implies periodic oscillation. Therefore, the torsion tensor P^i_{jk} expresses the aperiodic reversals of the magnetic field.

5. Discussions and conclusion

5.1. Relationship between geometrical objects and the magnetic field

In this section, the relation between geometrical objects and the behaviour of the magnetic field is investigated. As mentioned in section 2, the curvatures P^i_{jkl}, D^i_{jkl} and the torsion P^i_{jk} survive in the Berwald space. On the other hand, as mentioned in section 4, the Douglas tensor as curvature tensor D^i_{jkl} disappears in the Rikitake system. Therefore, two geometrical objects, i.e. the curvature P^i_{jkl} (or P^i_l) and the torsion P^i_{jk}, express the states of the Rikitake system. Moreover, the torsion tensor expresses the deviation curvature as follows:

$$P^i_j=P^i_{kj}y^k+P^i_{0j}, \tag{63}$$

where

$$P^i_{0j}=\epsilon^i_{|j}=\tfrac{1}{2}\left(\Gamma^m_{jk}\gamma^i_m-\Gamma^i_{mj}\gamma^m_k\right)y^k-\tfrac{1}{4}\gamma^i_m\gamma^m_j-f^m\Gamma^i_{mj}. \tag{64}$$

From this relation, it can be seen that the torsion tensor influences the Jacobi stability for the deviated trajectories. In the Rikitake system, the torsion tensor is geometrically a more fundamental object than the curvature tensor. Therefore, we first discuss the influence of the torsion tensor on the chaotic behaviour of the Rikitake system.

Let us investigate the components of the torsion tensor. From (62), the torsion tensor with respect to the current I^1, P^1_{jk}, is

$$P^1_{12}=-\Gamma^1_{23}N^3_1=\frac{1}{4}\frac{M^1_{23}}{L^1_1}\frac{M^3_{12}}{J^3_3}I^2, \qquad P^1_{13}=-\Gamma^1_{23}N^2_1=-\frac{1}{4}\frac{M^1_{23}}{L^1_1}\frac{M^2_{14}}{L^2_2}\omega^2, \tag{65}$$

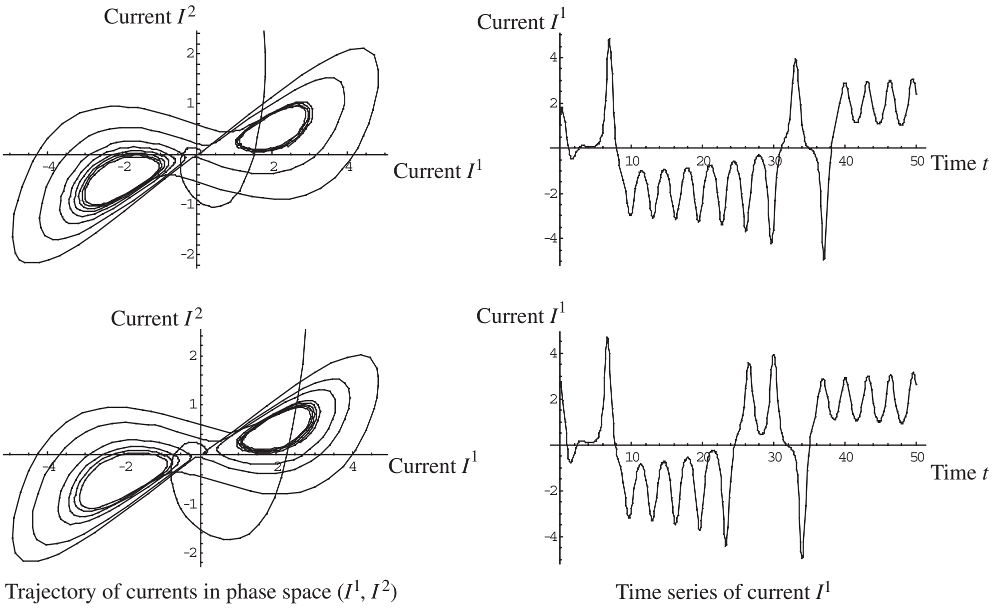

Figure 3. Numerical results for different initial conditions. $\mu = 1.2$, and $\nu = 4.5$. Top: $(y^1(0), y^2(0), y^3(0)) = (1.0, 4.0, 2.1)$ and bottom: $(2.5, 4.0, 2.1)$.

$$P^1_{23} = -\Gamma^1_{23} N^2_2 = \frac{1}{4}\frac{M^1_{23}}{L^1_1}\frac{R^2_2}{L^2_2}, \qquad P^1_{43} = -\Gamma^1_{23} N^2_4 = -\frac{1}{4}\frac{M^1_{23}}{L^1_1}\frac{M^2_{41}}{L^2_2} I^1. \tag{66}$$

Similarly, $P^2_{jk} \propto \Gamma^2_{14}$, $P^3_{jk} \propto \Gamma^3_{12}$ and $P^4_{jk} \propto \Gamma^4_{12}$. The torsion tensor P^1_{jk} can be expressed by the mutual-inductance Γ^1_{23} as the interaction between electrical field (y^2) and mechanical field (y^3), i.e. the interaction between the poloidal current field I^2 and the intensity of the α-effect ω^1. Moreover, the components of the torsion tensor P^1_{jk} can be expressed by the nonlinear connection N^i_j as the projection of the (y^i)-field into the (y^j)-field. For example, in P^1_{12}, the poloidal current I^2 and the α-effect ω^1 interact with each other. Then, the α-effect ω^1 also interacts with the toroidal current I^1. These two interactions, M^i_{jh} and N^h_k, thus express the torsion tensor P^i_{jk} and cause the chaotic behaviour of the Rikitake system.

Next, let us consider the effect of the torsion tensor on the second invariant P^i_l. Using the dimensionless form (30), we obtain the electrical deviation tensor:

$$4P^r_l = \begin{pmatrix} -(y^3)^2 + (y^2)^2 - \mu^2 & -y^1y^2 + 2\mu y^3 + 2 \\ -y^1y^2 + 2\mu y^3 - 2\mu\nu + 2 & -(y^3)^2 + (y^1)^2 + \nu y^3 - \mu^2 \end{pmatrix}. \tag{67}$$

The deviation curvature tensor determines the behaviour of trajectories when the initial conditions are varied. For example, let us consider the case when $\mu = 1.2, \nu = 4.5$ and the initial condition changes from $(y^1(0), y^2(0), y^3(0)) = (1.0, 4.0, 2.1)$ to $(y^1(0), y^2(0), y^3(0)) = (2.5, 4.0, 2.1)$. From these and equation (30), the computed time series and trajectories of current I^1 and I^2 in phase space are shown in figure 3; the top figure corresponds to $(y^1(0), y^2(0), y^3(0)) = (1.0, 4.0, 2.1)$ and the bottom one to $(y^1(0), y^2(0), y^3(0)) = (2.5, 4.0, 2.1)$. In the case of $(y^1(0), y^2(0), y^3(0)) = (1.0, 4.0, 2.1)$, the real part of the eigenvalues of the deviation curvature tensor is positive $\lambda = 1.84$. Similarly, in the case of $(y^1(0), y^2(0), y^3(0)) = (2.5, 4.0, 2.1)$, the deviation curvature tensor has positive eigenvalues $\lambda_1, \lambda_2 = 0.90, 4.10$. Hence, from theorem 2, the deviation curvature tensor shows that field trajectories in figure 3 are Jacobi unstable.

Next, a special case of the deviation curvature tensor is considered, i.e. when there is no torsion tensor P^i_{jk} or interaction M^i_{jk}. In this case, from relation (63), the deviation curvature tensor P^i_j is determined by only P^i_{0j}. Then, the deviation curvature tensor has the eigenvalues as $\lambda = -\left(R^1_1\right)^2/4, -\left(R^2_2\right)^2/4 < 0$ and $\lambda = 0$, i.e. the trajectories of the system are Jacobi stable. Therefore, the existence of the torsion tensor determines the instability of the deviated trajectories. Hence, the torsion tensor can be expressed by the mutual-inductances as the interaction between mechanical and electrical systems which causes the chaotic behaviour of the Rikitake system.

5.2. *Chaotic behaviour of the Rikitake system and the topological invariant*

By a topological invariant or magnetic helicity, the chaotic behaviour of the Rikitake system can be related to a magnetohydrodynamic motion in dynamo action.

The magnetic helicity of field **B** in K flux tubes with a vector potential **A** is defined by [8]:

$$\begin{aligned} H(\mathbf{B}) &= \int_V \mathbf{A}\cdot\mathbf{B}\,\mathrm{d}V \\ &\approx \sum_{i=1}^{K} \mathscr{T}_i(\Phi^i)^2 + \sum_{i,j=1}^{K} \mathscr{L}_{ij}\Phi^i\Phi^j, \end{aligned} \tag{68}$$

where Φ^i is the magnetic flux and V is a domain in $\mathbf{R}^3$. $\mathscr{T}_i$ is the self-helicity of a flux tube i and $\mathscr{L}_{ij}$ is the mutual-helicity of flux tubes i and j. By regarding the potential **A** and the field **B** as a connection 1-form Γ and a curvature 2-form $\Gamma\wedge\Gamma$, the magnetic helicity can be expressed by the Chern–Simons number [20]:

$$\mathrm{CS} \approx \int \mathrm{tr}\left(\Gamma\wedge\mathrm{d}\Gamma + \frac{2}{3}\Gamma\wedge\Gamma\wedge\Gamma\right), \tag{69}$$

where the potential $\Gamma = (\Gamma^i_k) = (\Gamma^i_{jk}\,\mathrm{d}x^j)$ is given by the spatial components of the connection coefficient (49). The $\wedge$ is an exterior product. In the Rikitake system, the term $\Gamma\wedge\mathrm{d}\Gamma$ vanishes because the connection coefficients Γ^i_{jk} are all constant. On the other hand, the term $\mathrm{tr}(\Gamma\wedge\Gamma\wedge\Gamma)$ is

$$\mathrm{tr}(\Gamma\wedge\Gamma\wedge\Gamma) = 3\left(\Gamma^1_{23}\Gamma^3_{12}\Gamma^2_{41}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^4 - \Gamma^1_{32}\Gamma^2_{14}\Gamma^4_{21}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\right).$$

Hence, the helicity or the Chern–Simons number of the Rikitake system is

$$\begin{aligned} \mathrm{CS} &\approx 2\int \left(\Gamma^1_{23}\Gamma^3_{12}\Gamma^2_{41}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^4 - \Gamma^1_{32}\Gamma^2_{14}\Gamma^4_{21}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\right) \\ &= 2\int \frac{M^2_{14}}{L^1_1}\,\mathrm{d}x^1\wedge\frac{M^1_{23}}{L^2_2}\,\mathrm{d}x^2\wedge\left(\frac{M^3_{12}}{J^3_3}\,\mathrm{d}x^4 - \frac{M^4_{12}}{J^4_4}\,\mathrm{d}x^3\right), \end{aligned} \tag{70}$$

where the mutual-inductances express the helicity. As mentioned in 5.1, the mutual-inductances in the torsion tensor imply the interactions which cause a chaotic behaviour. Therefore, the helicity can be related to the chaotic behaviour of the Rikitake system. Moreover, the existence of helicity is determined by the mechanical parts, $M^3_{12}\,\mathrm{d}x^4/J^3_3$ and $M^4_{12}\,\mathrm{d}x^3/J^4_4$. Therefore, the helicity vanishes when the coefficients satisfy $M^3_{12} = M^4_{12}$, $J^3_3 = J^4_4$ and the two discs rotate with similar angular velocities, $\omega^1 = \omega^2$ or $\mathrm{d}x^3 = \mathrm{d}x^4$. This state of non-helicity is equivalent to the non-chaotic behaviour of the Rikitake system because of the unique invariant $X = \{(y^1)^2 - (y^2)^2\}\exp(-2RLt)$ [24, 25, 37]. In magnetohydrodynamics, the existence of helicity expresses a turbulent state due to the α-effect [32]. Therefore, the helicity in equation (70) shows that the chaotic behaviour of the Rikitake system can be regarded as the turbulence motion in the magnetohydrodynamic dynamo.

In the case when the helicity does not vanish, the Chern–Simons number (70) can be integrated over the circuit of system I, II and the angle of disc, respectively;

$$\begin{aligned} \mathrm{CS} &= 2\int_{\mathrm{I}} \frac{M_{14}^2}{L_1^1}\,\mathrm{d}x^1 \int_{\mathrm{II}} \frac{M_{23}^1}{L_2^2}\,\mathrm{d}x^2 \left(\int_0^{2\pi} \frac{M_{12}^3}{J_3^3}\,\mathrm{d}x^4 - \int_0^{2\pi} \frac{M_{12}^4}{J_4^4}\,\mathrm{d}x^3\right) \\ &= \frac{4\pi}{L_1^1 L_2^2}\left(\frac{M_{12}^3}{J_3^3} - \frac{M_{12}^4}{J_4^4}\right) M_{14}^2 Q^1 M_{23}^1 Q^2 \\ &= \frac{4\pi T^2}{L_1^1 L_2^2}\left(\frac{M_{12}^3}{J_3^3} - \frac{M_{12}^4}{J_4^4}\right)\tilde{\Phi}^1\tilde{\Phi}^2 \\ &= \mathscr{L}_{12}\tilde{\Phi}^1\tilde{\Phi}^2, \end{aligned} \tag{71}$$

where Q^i is the total electric charge of q^i and $\tilde{\Phi}^i$ is the magnetic flux through the disc per unit time T, i.e. $\tilde{\Phi}^1 = M_{14}^2 Q^1/T = M_{14}^2 I^1$. From expressions (68) and (71), the mutual helicity is given by $\mathscr{L}_{12}$. The mutual-inductance M_{12}^3 gives the interaction between the toroidal current I^1 and the poloidal current I^2 with respect to the α-effect as ω^1. On the other hand, the mutual-inductance M_{12}^4 gives the interaction between the toroidal current I^1 and the poloidal current I^2 with respect to the ω-effect as ω^2. Thus, in the Rikitake system, the chaotic behaviour as a turbulent motion can be induced by the difference in the intensity of the interaction between poloidal and toroidal currents in the α- and ω-effects.

5.3. Other nonlinear systems and the KCC-theory

Finally, it is shown that the KCC-theory can be applied to other nonlinear dynamical systems.

In general, nonlinear systems can be unified into a single expression [38]:

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = A_{ij}y_j + B_{ijk}y_jy_k, \tag{72}$$

where A_{ij}, B_{ijk} are arbitrary functions and $i = 1, 2, 3$. The dynamical system (72) expresses a special case of the Rikitake system when $A_{ij} = \gamma_{ij}$, $B_{ijk} = G_{ijk}$ and $f_i = 0$ in equation (42). This system (72) expresses various nonlinear phenomena occurring in physics, chemistry and biology. For example, the equations of motion (72) express the Lorentz model in meteorology. The geometrization of the Lorentz model has been studied in terms of the Finsler geometry [9, 16]. The system (72) also expresses the Lotka–Volterra model in biology. The geometrization of the general case of the Lotka–Volterra model (Volterra–Hamilton system) has been studied by the KCC-analysis [2, 4–6]. Other areas where this dynamical system (72) appears are in plasma physics [17] and in Belousov-Zhabotinskii reaction model in chemistry [41]. In these dynamical systems, there exist five KCC-invariants and a topological invariant (Chern–Simons number) as in the case of the Rikitake system. Therefore, the five KCC-invariants and the topological Chern–Simons number can express the chaotic behaviour of the various nonlinear dynamical systems (72).

5.4. Conclusion

In this paper, the Rikitake system is studied from the view point of differential geometry (theory of Kosambi–Cartan–Chern). The electromechanical equations of motion derived from Faraday's law correspond to the magnetohydrodynamic equations. By applying the geometric theory to the Rikitake system, the behaviour of the system can be expressed by the five geometrical invariants. The third invariant is a torsion tensor which is made up of mutual-inductances. Therefore, the torsion tensor expresses the physical interaction between the electrical and mechanical systems. This interaction causes aperiodic reversals of the magnetic field. Moreover, a topological invariant, the Chern–Simons number, shows that the chaotic behaviour corresponds to a magnetohydrodynamical turbulent motion. This geometric theory can also be applied to other nonlinear dynamical systems. Thus, it is possible to analyse chaotic behaviour of the nonlinear dynamical systems based on such geometrical and topological invariants.

Acknowledgments

The authors would like to thank Professor Bender, Editor-in-Chief, and two anonymous referees for their helpful and valuable comments on our paper. The authors also acknowledge D Nair for revising the English style of our paper. One of the authors (T Yajima) is financially supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists and by the 21st Center-Of-Excellence program, 'Advanced Science and Technology Center for the Dynamic Earth', of Tohoku University.

References

[1] Allan D W 1962 On the behaviour of systems of coupled dynamos *Proc. Camb. Phil. Soc.* **58** 671–93

[2] Antonelli P L, Bevilacqua L and Rutz S F 2003 Theories and models in symbiogenesis *Nonlinear Anal.: Real World Appl.* **4** 743–53

[3] Antonelli P L and Bucătaru I 2001 New results about the geometric invariants in KCC-theory *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **47** 405–20

[4] Antonelli P L and Bucătaru I 2001 Volterra-Hamilton production models with discounting: general theory and worked examples *Nonlinear Anal.: Real World Appl.* **2** 337–56

[5] Antonelli P L, Ingarden R S and Matsumoto M 1993 *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology* (Dordrecht: Kluwer)

[6] Antonelli P L, Rutz S F and Sabău V S 2002 A transient-state analysis of Tyson's model for the cell division cycle by means of KCC-theory *Open Syst. Inf. Dyn.* **9** 223–38

[7] Berwald L 1947 Ueber Systeme von gewöhnlichen Differentialgleichungen zweiter Ordnung deren Integralkurven mit dem System der geraden Linien topologisch aequivalent sind *Ann. Math.* **48** 193–215

[8] Berger M A and Field G B 1984 The topological properties of magnetic helicity *J. Fluid Mech.* **147** 133–48

[9] Bordag L A and Dryuma V S 1997 Investigation of dynamical systems using tools of the theory of invariants and projective geometry *Z. Angew. Math. Phys.* **48** 725–43

[10] Bringuier E 2003 Electrostatic charges in $v \times B$ fields and the phenomenon of induction *Eur. J. Phys.* **24** 21–9

[11] Bullard E 1955 The stability of a homopolar dynamo *Proc. Camb. Phil. Soc.* **51** 744–60

[12] Cartan E 1933 Observations sur le mémoire précédent *Math. Z.* **37** 619–22

[13] Chern S S 1939 Sur la géométrie d'un système d'équations différentielles du second ordre *Bull. Sci. Math.* **63** 206–12

[14] Cook A E and Roberts P H 1970 The Rikitake two-disc dynamo system *Proc. Camb. Phil. Soc.* **68** 547–69

[15] Douglas J 1928 The general geometry of paths *Ann. Math.* **29** 143–68

[16] Dryuma V S 1994 Geometrical properties of the multidimensional nonlinear differential equations and the Finsler metrics of phase spaces of dynamical systems *Theor. Math. Phys.* **99** 555–61

[17] Fuchs V 1975 The influence of linear damping on nonlinearly coupled positive and negative energy waves *J. Math. Phys.* **16** 1388–92

[18] Hide R 1995 Structural instability of the Rikitake disk dynamo *Geophys. Res. Lett.* **22** 1057–9

[19] Ito K 1980 Chaos in the Rikitake two-disc dynamo system *Earth Planet. Sci. Lett.* **51** 451–6

[20] Jackiw R and Pi S-Y 2000 Creation and evolution of magnetic helicity *Phys. Rev.* D **61** 105015

[21] Kondo K 1955 Geometry of paths as applied to the theory of dynamical systems *RAAG Memoirs of the Unifying Study of the Basic Problems in Engineering Sciences by Means of Geometry* vol I, ed K Kondo (Tokyo: Gakujutu Bunken Fukyu-Kai) pp 316–34

[22] Kosambi D D 1933 Parallelism and path-spaces *Math. Z.* **37** 608–18
[23] Kron G 1934 Non-Riemannian dynamics of rotating electrical machinery *J. Math. and Phys.* **13** 103–94
[24] Labrunie S and Conte R 1996 A geometrical method towards first integrals for dynamical systems *J. Math. Phys.* **37** 6198–206
[25] Llibre J and Zhang X 2000 Invariant algebraic surfaces of the Rikitake system *J. Phys. A: Math. Gen.* **33** 7613–35
[26] Miron R and Frigioiu C 2005 Finslerian mechanical systems *Algebras Groups Geom.* **22** 151–67
[27] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (New York: Freeman)
[28] Moffatt H K 1978 *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge: Cambridge University Press)
[29] Montgomery H 1999 Unipolar induction: a neglected topic in the teaching of electromagnetism *Eur. J. Phys.* **20** 271–80
[30] Montgomery H 2004 Current flow patterns in a Faraday disc *Eur. J. Phys.* **25** 171–83
[31] Moroz I M, Hide R and Soward A M 1998 On self-exciting coupled Faraday disk homopolar dynamos driving series motors *Physica* D **117** 128–44
[32] Parker E N 1955 Hydromagnetic dynamo models *Astrophys. J.* **122** 293–314
[33] Rikitake T 1958 Oscillations of a system of disk dynamos *Proc. Camb. Phil. Soc.* **54** 89–105
[34] Sabău V S 2005 Systems biology and deviation curvature tensor *Nonlinear Anal.: Real World Appl.* **6** 563–87
[35] Schouten J A 1989 *Tensor Analysis for Physicists* (New York: Dover)
[36] Steeb W-H 1982 Classical mechanics and constants of motion *Hadronic J.* **5** 1738–47
[37] Steeb W-H 1982 Continuous symmetries of the Lorenz model and the Rikitake two-disc dynamo system *J. Phys. A: Math. Gen.* **15** L389–90
[38] Steeb W-H, Kunick A and Strampp W 1983 The Rikitake two-disc dynamo system and the Painlevé property *J. Phys. Soc. Japan* **52** 2649–53
[39] Woodson H H and Melcher J R 1968 *Electromechanical Dynamics: Part I. Discrete Systems* (New York: Wiley)
[40] Yano K 1955 *The Theory of Lie Derivatives and Its Applications* (Amsterdam: North-Holland)
[41] Zhabotinskii A M 1964 Periodic oxidising reactions in the liquid phase (in Russian) *Dokl. Akad. Nauk. SSSR* **157** 392–5 (Belousov 1958 cited therein)